

Lines and Planes

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Equation of Lines

Equations of Lines

An **equation of a line** can be obtained if you have a point and a direction. Given a point $\vec{r}_0 = (x_0, y_0, z_0)$ and a direction $\vec{d} = \langle a, b, c \rangle$, and equation of a line in **vector form** can be written as

$$\vec{r} = \vec{r}_0 + \vec{d}t = (x_0, y_0, z_0) + \langle a, b, c \rangle t$$

where t is a parameter.

By gathering the components, we can rewrite the equation in **parametric form**:

$$x(t) = at + x_0, \quad y(t) = bt + y_0, \quad z(t) = ct + z_0$$

If we solve for t and set them equal to each other, we get the equation in **symmetric form**:

$$x = at + x_0 \rightarrow t = \frac{x - x_0}{a}, \quad y = bt + y_0 \rightarrow t = \frac{y - y_0}{b}, \quad z = ct + z_0 \rightarrow t = \frac{z - z_0}{c}$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Note that these equations are *not unique*.

Example 1:

Find an equation of the line passing through the points $P(4,0,5)$ and $Q(2,3,1)$

Handwritten solution on blue grid paper:

$$P(4,0,5) \quad Q(2,3,1)$$
$$(2-4, 3-0, 1-5)$$
$$(-2, 3, -4) \leftarrow \text{Direction}$$
$$x = 4 - 2t$$
$$y = 0 + 3t$$
$$z = 5 - 4t$$

Distance Between a Point and a Line

Distance between a point and a line:

The idea:

- Find the vector connecting the point and the line.
- Find the area of the parallelogram spanned by this vector and the direction of the line.
- Area of this parallelogram is the base times the height (distance). Divide by the base...

Let L be the line in space passing through the point P with direction vector $\vec{d}$. Let M be any other point not on the line L . Let $\vec{v} = \overrightarrow{PM}$. Then the distance from M to L is

$$dist = \frac{\|\vec{v} \times \vec{d}\|}{\|\vec{d}\|}$$

Example 2:

Find the distance between the point $(4,3,5)$ and the line $x = -1 - t, y = -t, z = 2$.

Handwritten solution on lined paper:

$$\begin{aligned} & \frac{1-10}{6} = \frac{6-10}{6} \\ & \langle 4, 3, 5 \rangle - \langle -1, 0, 2 \rangle = \langle 5, 3, 3 \rangle \\ & \begin{array}{ccc} 5 & -1 & 3, -3, -2 \\ 3 & -1 & \\ 3 & 0 & \end{array} \quad \frac{\sqrt{9+9+2}}{\sqrt{2}} = \frac{\sqrt{22}}{\sqrt{2}} \\ & \qquad \qquad \qquad = \sqrt{11} \\ & 3(0) - 3(-1) = 3 \\ & 3(-1) - 5(0) = -3 \\ & 5(-1) - 3(-1) = -2 \end{aligned}$$

Two Lines in 3D

Two Lines

Two different lines either (1) are parallel, (2) intersect at exactly one point, or (3) are **skew**.

- (1) Lines are parallel if the direction vectors are parallel.
- (2) Lines intersect if we can find values for two (possibly different parameters) that make the components of the line equal.
- (3) If they are not parallel and they do not intersect, then the lines are skew.

Example 3a:

$$\vec{r} = \langle 3, 3, 0 \rangle + t \langle 3, -3, 2 \rangle \text{ and } \vec{r} = \langle 6, 0, 2 \rangle + t \langle -3, 3, 0 \rangle$$

Direction

$$\vec{r} = \langle 3, 3, 0 \rangle + t \langle 3, -3, 2 \rangle$$
$$\vec{r} = \langle 6, 0, 2 \rangle + t \langle -3, 3, 0 \rangle$$

not parallel, slopes are different

$$\begin{aligned} 3+3t &= 6-3s \\ 3t+3s &= 3 \\ t+s &= 1 \end{aligned}$$
$$\begin{aligned} 3-3t &= 0+3s \\ 3t-3s &= 3 \\ t-s &= 1 \end{aligned}$$

means intersect

$$\begin{aligned} 2+0s &= 0+2t \\ 2t &= 2 \\ t &= 1 \end{aligned}$$

plug [1] into equation

$$\begin{aligned} 3+3 &= 6 \\ 3-3 &= 0 \\ 0+2 &= 2 \end{aligned}$$

intersect

Insert CalcPlot3D link here: [link](#)

Example 3b:

Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting:

$$L_1: \frac{x-3}{2} = 2-y = \frac{z-1}{3}, \quad L_2: \frac{x-2}{4} = \frac{1-y}{2} = \frac{z-2}{3}$$

Handwritten calculations on blue grid paper:

Top section (for L_1):

$$\begin{aligned} x-3 &= 2-y \\ y &= 2-x+3 \\ y &= 5-x \end{aligned}$$

$$\frac{x-3}{2} = 2-y = \frac{z-1}{3}$$

$$\frac{x-3}{2} = 2-(5-x) = x-3$$

$$\frac{x-3}{2} = x-3 \implies x-3 = 2x-6 \implies x = 3$$

$$y = 5-3 = 2$$

$$\frac{z-1}{3} = 2-2 = 0 \implies z-1 = 0 \implies z = 1$$

Bottom section (for L_2):

$$\frac{x-2}{4} = \frac{1-y}{2} = \frac{z-2}{3}$$

$$\frac{x-2}{4} = \frac{1-y}{2} \implies x-2 = 2(1-y) = 2-2y \implies x = 4-2y$$

$$\frac{x-2}{4} = \frac{z-2}{3} \implies 3(x-2) = 4(z-2) \implies 3x-6 = 4z-8 \implies 3x = 4z-2$$

$$3(4-2y) = 4z-2 \implies 12-6y = 4z-2 \implies 14 = 4z+6y \implies 7 = 2z+3y$$

$$y = \frac{7-2z}{3}$$

$$x = 4-2\left(\frac{7-2z}{3}\right) = 4-\frac{14-4z}{3} = \frac{12-14+4z}{3} = \frac{-2+4z}{3}$$

Conclusion:

For L_1 , $x=3, y=2, z=1$

For L_2 , $x = \frac{-2+4z}{3}, y = \frac{7-2z}{3}, z = z$

Since the lines are not parallel and intersect at $(3, 2, 1)$, they are intersecting lines.

Insert CalcPlot3D link here: [link](#)

Equations of Planes

Planes

A **plane** in $\mathbb{R}^3$ can be uniquely defined by a point and a normal vector.

Suppose $\vec{r}_0 = (x_0, y_0, z_0)$ is a point on a plane and $\vec{N} = \langle A, B, C \rangle$ is a vector perpendicular to the plane, then all the points $\vec{r} = (x, y, z)$ on the plane can be described (in **vector form**) as

$$\vec{N} \cdot (\vec{r} - \vec{r}_0) = 0, \quad \vec{N} \cdot \vec{r} = \vec{N} \cdot \vec{r}_0$$

By using the components:

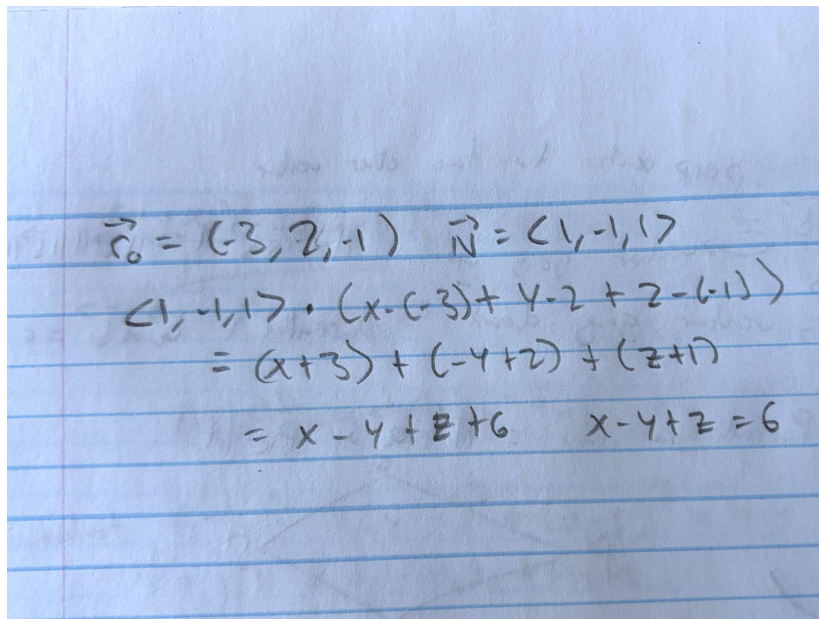
$$\langle A, B, C \rangle \cdot (x, y, z) = \langle A, B, C \rangle \cdot (x_0, y_0, z_0)$$

We then let $D = \langle A, B, C \rangle \cdot (x_0, y_0, z_0)$, so we can express the equation in **scalar form**:

$$Ax + By + Cz = D$$

Example 4a:

Find an equation of a plane passing through the point $(-3, 2, -1)$ with a normal vector $\langle 1, -1, 1 \rangle$.



Handwritten solution on lined paper:

$$\begin{aligned}\vec{r}_0 &= (-3, 2, -1) \quad \vec{N} = \langle 1, -1, 1 \rangle \\ \langle 1, -1, 1 \rangle \cdot (x - (-3)) + y - 2 + z - (-1) &= 0 \\ &= (x + 3) + (-y + 2) + (z + 1) \\ &= x - y + z + 6 \quad x - y + z = -6\end{aligned}$$

Example 4b:

Find an equation of a plane passing through the origin and the points $P(4, -1, 5)$ and $Q(5, 1, 2)$

4b. $\vec{n} = \vec{v} \times \vec{w}$

4	5	$-2 - 5 = -7$
-1	1	$25 - 8 = 17$
5	2	$4 - (-5) = 9$

$\vec{n} = \langle -7, 17, 9 \rangle$

$$-7(x-0) + 17(y-0) + 9(z-0) = 0$$
$$-7x + 17y + 9z = 0$$

Intersection between a Line and a Plane

To find the intersections between a line and a plane...

Write the line in parametric form and plug the expressions into the x , y , and z variables in the plane equation. IF the line intersects the plane, the resulting equation should be a linear equation that can easily be solved. If there is NO solution, then the line does not intersect the plane.

Examples 5a:

Find the point at which the line $\langle 4 - t, 2 + t, 4t \rangle$ intersects the plane $x - y + 2z = 8$.

Handwritten solution on blue grid paper:

$$\begin{aligned}x - y + 2z &= 8 \\2z &= -x + y + 8 \\z &= \frac{-x + y}{2} + 4\end{aligned}$$

Substituting the parametric equations of the line $x = 4 - t$, $y = 2 + t$, $z = 4t$ into the plane equation:

$$(4 - t) - (2 + t) + 2(4t) = 8$$
$$4 - t - 2 - t + 8t = 8$$
$$2 + 6t = 8$$
$$6t = 6$$
$$t = 1$$

Substituting $t = 1$ back into the parametric equations of the line:

$$\begin{aligned}x &= 4 - 1 = 3 \\y &= 2 + 1 = 3 \\z &= 4(1) = 4\end{aligned}$$

The intersection point is $(3, 3, 4)$.

Examples 5b:

Find the point of intersection between the line and the plane:

$$\vec{r}(t) = (0,0,2) + \langle 1, -1, 0 \rangle t, \quad x + y + z = 1$$

Handwritten solution on blue grid paper:

$$\begin{aligned} x &= t \\ y &= -t \\ z &= 2 \end{aligned}$$

Sub $\rightarrow$

$$\begin{aligned} (+) + (-t) + (2) &= 1 \\ + -t + 2 &= 1 \\ 2 &= 1 \end{aligned}$$

the line does not intersect with the plane

A large arrow points from the conclusion to the equations above it.

Intersection between Two Planes

To find the intersection between two planes,

- Find the normal vectors for each plane.
- Compute the cross product of the two normal vectors:
 - The resulting vector should be perpendicular to both normals, and...
 - This vector is on both planes, and hence will be the direction vector of the line of intersection.
- In order to find a point,
 - set one of the variables to be a constant (for example, $z=0$).
 - Then you will have two equations and two unknowns. Solve for the other two variables and...
 - you will have your point that lie on both planes... and hence the intersection.
- Use the point and the direction vector to find an equation of the line.

Examples 6:

Find the line of intersection between the planes $x + y - z = 3$ and $2x - y + 3z = 1$.

Handwritten solution on lined paper:

$$\begin{aligned} 6. \quad & x + y - z = 3 \quad 2x - y + 3z = 1 \\ & \begin{array}{r} 1 \quad 2 \quad 3-1 = 2 \\ 1 \quad -1 \quad -2-3 = -5 \\ -1 \quad 3 \quad -1-2 = -3 \end{array} \quad \vec{v} = \langle 2, -5, -3 \rangle \\ & x + y = 3 \quad x = \frac{4}{3} \quad \frac{4}{3} + \frac{5}{3} - z = 3 \\ & 2x - y = 1 \quad \frac{4}{3} + y = 3 \quad z = 0 \\ & 3x = 4 \quad y = \frac{5}{3} \\ & x = 2t + \frac{4}{3} \quad y = -5t + \frac{5}{3} \quad z = -3t \end{aligned}$$